

Early Journal Content on JSTOR, Free to Anyone in the World

This article is one of nearly 500,000 scholarly works digitized and made freely available to everyone in the world by JSTOR.

Known as the Early Journal Content, this set of works include research articles, news, letters, and other writings published in more than 200 of the oldest leading academic journals. The works date from the mid-seventeenth to the early twentieth centuries.

We encourage people to read and share the Early Journal Content openly and to tell others that this resource exists. People may post this content online or redistribute in any way for non-commercial purposes.

Read more about Early Journal Content at http://about.jstor.org/participate-jstor/individuals/early-journal-content.

JSTOR is a digital library of academic journals, books, and primary source objects. JSTOR helps people discover, use, and build upon a wide range of content through a powerful research and teaching platform, and preserves this content for future generations. JSTOR is part of ITHAKA, a not-for-profit organization that also includes Ithaka S+R and Portico. For more information about JSTOR, please contact support@jstor.org.

It is clear that $\frac{1}{n}$ of any magnitude may be repeated as a unit just as well as $\frac{n}{n}$ or $\frac{3n}{n}$; it is equally plain that $\frac{m}{n}$ is as much an expression of ratio as is m. Hence each definition applies to fractions as well as integers.

It is neither necessary nor advisable to divide ("break") single things (individuals, as apples) into parts in order to get fractions. In counting the eggs in a dozen (e. g.) the wee bairn is on the border of the fairyland of fractions, though he may not be conscious of it. At any stage of his counting the result is either integral or fractional. Five eggs is integral with respect to the unit (1 egg); it is fractional with respect to the unity or whole (dozen)—5 out of 12, 5 twelfths. Five half-yards is just as integral as 5 yards. The ratio in each is five. But in $\frac{5}{2}$ yards the ratio is $\frac{5}{2}$; the fractional idea is present, owing to the denominator, which defines the unit of measure.

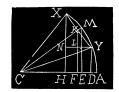
SOME TRIGONOMETRIC RELATIONS PROVED GEOMETRICALLY.

By P. H. PHILBRICK, C. E., Pineville, Louisiana.

Most trigonometric formulæ may be proven geometrically in an elegant manner; and moreover, the relations between the trigonometric functions may be shown at a glance by means of the geometric figures. The results are all the more interesting, too, when proven also directly from first principles. For this reason the following exercises are offered.

For convenience, describe the arc AYX, and take the radius AC for the unit of measurement. Let the arc AX=x and arc AY=y. Take M at the middle of XY, and draw lines as indicated.

Then $DY=\sin y$, $HX=\sin x$, $EM=\sin \frac{1}{2}(x+y)$, $KY=\sin \frac{1}{2}(x-y)$, $NX=\sin x-\sin y$, $NY=\cos y-\cos x$, $CE=\cos \frac{1}{2}(x+y)$, $CK=\cos \frac{1}{2}(x-y)$.



Now,
$$HX+DY=2KF=2EM\frac{KF}{EM}=2EM\frac{CK}{CM}=2EM.CK$$
.

That is,
$$\sin x + \sin y = 2\sin \frac{1}{2}(x+y)\cos \frac{1}{2}(x-y)$$
....(1).

by Drs. McClellan and Dewey. It is interesting in matter, vigorous and aggressive in style, refreshing in its originality, and scholarly in its conception and execution. It is in the 33d volume of the International Education Series, published by D. Appleton & Co., New York.

Again,
$$CH + CD = 2CF = 2CE \cdot \frac{CF}{CE} = 2CE \cdot \frac{CK}{CM} = 2CE \cdot CK$$
,

or
$$\cos x + \cos y = 2\cos \frac{1}{2}(x+y)\cos \frac{1}{2}(x-y)$$
....(2).

The triangles CEM and XNY are similar;

hence
$$\frac{NX}{NY} = \frac{CE}{CM}$$
, or $NX = 2CE + \frac{\frac{1}{2}XY}{CM} = 2CE.KY$,

that is, $\sin x - \sin y = \cos \frac{1}{2}(x+y)\sin \frac{1}{2}(x-y)$ (3).

Similarly,
$$\frac{NY}{XY} = \frac{EM}{CM}$$
, or $NY = 2EM \frac{1}{2}XY = 2EM.KY$,

Equation (1) can be made very useful in computing trigonometric tables, as the writer intends subsequently to show.

Now let AM=x and MY=MX=y. Then AY=x-y and AX=x+y.

We have
$$(CM)^2 - (CK)^2 = (CY)^2 - (CK)^2 = (KY)^2$$
. But $\frac{CM}{ME} = \frac{CK}{KF} = \frac{KY}{LY}$.

Therefore
$$(ME)^2 - (KF)^2 = (LY)^2 = (KY)^2 - (KL)^2$$
,

or
$$(KF)^2 - (KL)^2 = (ME)^2 - (KY)^2$$
,

or
$$(KF+KL)(KF-KL)=(ME)^2-(KY)^2$$
,

or
$$HX \times DY = (ME)^2 - (KY)^2$$
.

That is, $\sin(x+y)\sin(x-y) = \sin^2 x - \sin^2 y = \cos^2 y - \cos^2 x$(5).

Again,
$$\frac{CM}{CE} = \frac{CK}{CF} = \frac{KY}{KL}$$
.

Therefore $(CE)^2 - (CF)^2 = (KL)^2 = (KY)^2 - (LY)^2$,

or
$$(CF)^2 - (LY)^2 = (CE)^2 - (KY)^2$$
,

or
$$(CF-LY)(CF+LY) = CH \times CD = (CE)^2 - (KY)^2$$
.

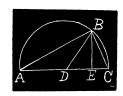
That is, $\cos(x+y)\cos(x-y) = \cos^2 x - \sin^2 y = \cos^2 y - \sin^2 x \dots (6)$.

Let DC=R the radius of a circle. Let the angle CDB=2x. Then DAB=DBA=CBE=x.

Then we have $\tan x = \frac{EC}{EB}$, also $\tan x = \frac{BE}{AE}$.

The product of these gives, $\tan^2 x = \frac{CE}{AE}$, or $CE \times AE = (BE)^2$,

or
$$\frac{EC}{AE} = \left(\frac{BE}{AE}\right)^2 = \tan^2 x$$
.



Also,
$$\frac{EC}{BE} = \frac{\text{vers}2x}{\sin 2x} = \frac{1 - \cos 2x}{\sin 2x} = \frac{\sin 2x}{1 + \cos 2x} = \tan x \text{ [see above]} \dots (7).$$

Then
$$1 + \tan^2 x = 1 + \frac{EC}{AE} = \frac{AC}{AE} = \frac{2R}{AE}$$

$$1 - \tan^2 x = 1 - \frac{EC}{AE} = \frac{AE - EC}{AE} = \frac{AC - 2EC}{AE} = \frac{2(R - EC)}{AE}$$
.

 $\cot 2x = \frac{DE}{RE}$ and $\csc 2x = \frac{R}{RE}$. From these values we at once have,

$$\frac{2\tan x}{1+\tan^2 x} = \frac{2BE}{A} \frac{AE}{2R} = \frac{BE}{R} = \sin 2x \qquad (8).$$

$$\frac{2\tan x}{1-\tan^2 x} = \frac{2BE}{AE} \cdot \frac{AE}{2(R-EC)} = \frac{BE}{R-EC} = \frac{BE}{DE} = \tan 2x \quad ... \quad (9).$$

$$\tan^2 x + 2\cot 2x \tan x = \frac{EC}{AE} + \frac{2DE}{BE} \cdot \frac{BE}{AE} = \frac{EC + 2DE}{AE} = \frac{AE}{AE} = 1 \quad ... \quad (10).$$

$$2\cos \operatorname{ec2xtanx} - \tan^2 x = \frac{2R}{RE} \cdot \frac{BE}{AE} - \frac{EC}{AE} = \frac{2R - EC}{AE} = \frac{AE}{AE} = 1 \quad \dots \quad (11).$$

$$\frac{1-\tan^2 x}{1+\tan^2 x} = \frac{2(R-EC)}{AE} \cdot \frac{AE}{2R} = \frac{R-CE}{R} = \frac{ED}{R} = \cos 2x \cdot \dots (12).$$

$$\frac{1+\sin 2x-\cos 2x}{1+\sin 2x+\cos 2x} = \frac{R+BE-ED}{R} \div \frac{R+BE+ED}{R} = \frac{EC+BE}{AE+BE}.$$

But
$$AE = \frac{(BE)^2}{EC}$$
; $\therefore \frac{CE + BE}{AE + BE} = \frac{EC + BE}{(BE)^2 \div EC + BE}$

$$= \frac{EC(EC + BE)}{BE(EC + BE)} = \frac{EC}{BE} = \tan x. \tag{15}.$$

Again,
$$\cos x = \frac{AE}{AB}$$
, also $\cos x = \frac{AB}{AC}$.

Twice the product of these gives $2\cos^2 x = \frac{2AE}{AC} = \frac{AE}{R}$.

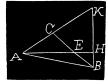
Also
$$\cos 2x = \frac{DE}{R}$$
. $1 + \cos 2x = \frac{DE + R}{R} = \frac{AE}{R}$. $\therefore 1 + \cos 2x = 2\cos^2 x \dots (16)$.

$$\sin x = \frac{CB}{AC} = \frac{BC}{2R}$$
; also $\sin x = \frac{EC}{BC}$. Twice the product of these gives

$$2\sin^2 x = \frac{EC}{R}$$
. $1 - \cos 2x = \frac{R - ED}{R} = \frac{EC}{R}$. $\therefore 1 - \cos 2x = 2\sin^2 x \dots (17)$.

To prove the "Tangent Proportion," let ABC be a plane triangle, the parts being represented as usual. Take CE=CA and draw AEH. Draw BHK

perpendicular to AH, to meet AC prolonged in K. Now considering the triangles ABC and ACE, the sum of the angles at A and E of the one is equal to the sum of the angles at A and B of the other. Hence CAE + CEA = A + B; and $CAE = CEA = BEH = \frac{1}{2}(A + B)$.



Also $BAE = A - \frac{1}{2}(A+B) = \frac{1}{2}(A-B)$. The angles at B and K of the triangle BCK are equal; for CBK is the complement of BEH or AEC, and BKC is the complement of the

complement of BEH or AEC, and BKC is the complement of the equal angle CAE. Hence CK=CB=a and AK=a+b.

Now
$$\tan \frac{1}{2}(A-B) = \frac{BH}{AH}$$
 and $\tan \frac{1}{2}(A+B) = \frac{HK}{AH}$. $\therefore \frac{\tan \frac{1}{2}(A-B)}{\tan \frac{1}{2}(A+B)} = \frac{BH}{HK}$.

But
$$\frac{BH}{HK} = \frac{BE}{AK} = \frac{a-b}{a+b}$$
. $\therefore \frac{\tan\frac{1}{2}(A-B)}{\tan\frac{1}{2}(A+B)} = \frac{a-b}{a+b}$(1).

From the triangle ABE,
$$\frac{BE}{AB} = \frac{\sin BAE}{\sin AEC}$$
, or $\frac{a-b}{c} = \frac{\sin \frac{1}{2}(A-B)}{\sin \frac{1}{2}(A+B)}$...(2).

In the triangle AHK, $AH=AK\cos HAK=(a+b)\cos \frac{1}{2}(A+B)$.

In the triangle ABH, $AH = AB\cos BAH = c\cos \frac{1}{2}(A-B)$.

Equating, we have,
$$\frac{a+b}{c} = \frac{\cos\frac{1}{2}(A-B)}{\cos\frac{1}{2}(A+B)}$$
(3).

Equation (3) divided by (2) also gives (1).